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# An Approximate Analytical Solution of Non-Linear Fractional-Order Constrained Optimization Problem Using Optimal Homotopy Analysis Method

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## Abstract

We present an optimal homotopy analysis method (OHAM) to find an accurate approximate analytic solution (AAS) for non-linear fractional-order constrained optimization problem (FOCOP). The previous analytical approximate method (AAM) of solving FOCOP possesses no norms for the convergence of the infinite series solution. OHAM provides an independent way of choosing proper values of the control-convergence parameter (CCP), auxiliary linear operator, and enables us to control and govern the convergence area of the series solution produced by a squared residual error optimization technique. Numerical comparisons of OHAM with Runge-Kutta fourth-order (RK4) method for accuracy. Some examples from the CUTEr library were used to indicate the correctness and relevance of the suggested techniques.

**Disciplinary**: Mathematics.



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## 1 Introduction

In optimization, different methods had been considered by several authors for solving constrained non-linear programming optimization problems in a class of integer-order systems of ordinary differential equations (ODEs). The gradient-based steepest descent approach is a solution method. The procedure converts a non-linear problem of optimization to systems of non-linear ODE dynamic with optimality requirements to get optimal solutions (Franceschi et. Al., 2017).

The computation of fractional-order began receiving much importance in the area of applied science some years back. Some authors in optimization focused on improving AAM for solving

various forms of arbitrary-order gradient-based dynamic systems from optimization problem: Such as a fractional dynamics trajectory approach (Evirgen and Özdemir, 2012), fractional steepest descent approach (Pu et al., 2013), an arbitrary gradient-based system using VIM (Evirgen, 2016), and a conformable fractional gradient-based system (Evirgen, 2017).

One-step OHAM was developed by Niu and Wang (2010) for non-linear differential equations(NLDEs). Liao (2010) proposed OHAM strong NLDEs to obtain an optimal convergence-control parameter by using optimization a method called squared residual error, which was integrated into the whole region of the governing equation for accurate optimal convergence-control parameters. OHAM approach has still not been implemented in solving non-linear FOCOP, which motivates this study.

## 2 Preliminaries

We begin by defining some particular functions and properties of fractional calculus (Abdeljawad, 2015).

**Definition 1**

Let $g: [0, \infty) \to \Re$ be a given function. The $\beta^{th}$ order conformable fractional derivative operator of $g$ given by

$$T^{\beta}(g)(x) = \lim_{\epsilon \to 0} \frac{g(x+\epsilon x^{1-\beta}) - g(x)}{\epsilon} \qquad (1),$$

$\forall\ x > 0$ and $\beta \in (0, 1]$.

**Theorem 1**

The function $\frac{d^{\beta} g}{dx^{\beta}} = x^{1-\beta} \frac{dg}{dx}$, if $g$ is differentiable, $0 < \beta \leq 1$, and $(f, g)$ be $\beta$-differentiable at a point $x > 0$.

**Definition 2**

Given that the integral is the regular Riemann improper, then we have

$I_a^{\beta}(g)(x) = I_a^1(x^{\beta-1} g) = \int_a^x \frac{g(t)}{t^{1-\beta}} dt$, where t, and $\beta \in (0, 1]$.

**Theorem 2**

$T^{\beta} I_a^{\beta}(g)(x) = g(x)$, given that $g$ is a continuous function in the domain of $I^{\beta}$ and $x \geq a$.

**Theorem 3**

Let $g: (a, b) \to \Re$ be differentiable and $0 < \beta \leq 1$. And so for all $x > a$, we have

$$I_a^{\beta} T^{\beta}(f)(x) = g(x) - g(a).$$

Taking non-linear programming, constrained optimization problem (NLPCOP) of the figure

$$\min_{x \in \Re^n} g(x) \quad subject \quad to \quad \psi_k(x) \leq 0 \ and \quad h_k(x) = 0 \ \forall \ k \in \Re. \qquad (2),$$

where $g: \Re^n \to \Re, \psi_k: \Re^n \to \Re$, and $h_k: \Re^n \to \Re$, $k \in \Re$, are $C^2$ functions. Let $X_0 = \{x \in \Re^n | h_k = 0, \psi_k \leq 0, k \in \Re\}$ be the feasible set of Equation (2), and $X_0$ is a set of functions. An efficient penalty function of Equation (2) is given as

$$P_{penalty}(h_k(x)) = \varrho \frac{1}{\sigma} \sum_{k=1}^{p} (h_k(x))^{\sigma}, \quad (3),$$

$$P_{penalty}(\psi_k(x)) = \varrho \sum_{k=1}^{p} (max\{0, \psi_k(x)\})^{\sigma}. \quad (4),$$

where $\sigma = 2$. It can be understood from previous research that the solution to Equation (2) is of the unconstrained form given as

$$min\ G(x, \varrho) = g(x) + \varrho(\frac{1}{\sigma} \sum_{k=1}^{p} (h_k(x))^{\sigma} + \sum_{k=1}^{p} (max\{0, \psi_k(x)\})^{\sigma}) \quad (5),$$

$$subject\ \ to\ \ x \in \Re^n$$

where $\varrho > 0$ is an auxiliary penalty variable (Nguyen et al., 2019).

Using Equation (1) and Equation (2) for problem Equation (5) with $\sigma = 2$, the conformable operator gradient-based dynamic model is formulated as

$$T^{\beta} x_t = -\nabla_x G(x, \varrho). \quad (6),$$

with the given initial conditions

$$x_k(0) = x_{k0} \qquad k = 1 \ldots . n. \quad (7),$$

where $\nabla_x G(x, \varrho)$ is the gradient vector and $x \in \Re$. This form of optimization problem solution was first introduced in Evirgen and Özdemir (2011).

Note that a point $x_e$ it is an equilibrium point of Equation (6) if it correspondent to the right-hand side of Equation (6). We reformulate the arbitrary-order dynamic system Equation (6) as

$$T^{\beta} x_k(t) = g_k(t, \varrho, x_1, x_2 \ldots . x_n), \ \ k = 1, 2 \ldots n \quad (8).$$

## 3 Methodology

We solve Equation (8) by constructing homotopy of the form

$$T^{\beta} x_k(t) = p g_k(t, \varrho, x_1, x_2 \ldots . x_n) \quad (9),$$

where $k = 1, 2 \ldots n$, and $p \in [0, 1]$. If $p = 0$, Equation (9) becomes

$$T^{\beta} x_k(t) = 0. \quad (10)$$

and when $p = 1$, the homotopy Equation (9)

$$T^{\beta} x_k(t) = g_k(t, \varrho, x_1, x_2 \ldots . x_n) \quad t \geq 0, \ \ 0 < \beta \leq 1. \quad (11)$$

with given initial conditions

$$x_k(0) = a_k, \ \ k = 1, 2 \ldots n \quad (12).$$

From (Liao, 2010), we formulate the zeroth-order deformation equations

$$(1 - q)\ell_k[T^{\beta} \phi_k(t, q) - x_{k0}(t)] = q\hbar_k[T^{\beta} \phi_k(t, q) - g_k(t, \varrho, \phi_1(t, q), \phi_2(t, q)\phi_n(t, q)]$$
$$k = 1, 2 \ldots . n \quad (13),$$

where $q \in [0,1]$ is an enclosed parameter, $\ell_k$ Are auxiliary linear operators satisfying $\ell_k(0) = 0$, $x_{k0}(t)$ are guessing approximation $x_k(t)$, $\hbar_k \neq 0$ are converging-control parameters, and $\phi_k(t, q)$ are unknown functions, when $q = 0$, and $q = 1$ we have

$$\phi_k(t,0) = x_{k0}(t), \qquad \phi_k(t,1) = x_k(t) k = 1,2,3\ldots n \tag{14}$$

Thus, as $q$ increasing from 0 *to* 1, the solution $\phi_k(t,q)$ ranges from the initial guess $x_{k0}(t)$ to the solutions $x_k(t)$. Expand $\phi_k(t,q)$ in a Taylor series for $q$, we have

$$\phi_k(t,q) = x_{k0}(t) + \sum_{m=1}^{\infty} x_{km}(t) q^m \tag{15},$$

where

$$x_{km}(t) = \frac{1}{m!} \frac{\partial^m \phi_k(t,q)}{\partial q^m}|_{q=0}, \tag{16}.$$

If the auxiliary linear operators $\ell_k$, Convergence-control parameters $\hbar_k$, And the guess approximation $x_{k0}(t)$, are correctly selected, then the series Equation (15) converges at $q = 1$, one has

$$x_k(t) = x_{k0}(t) + \sum_{m=1}^{\infty} x_{km}(t) \qquad k = 1,2,3\ldots n \tag{17}.$$

As proposed by Odibat (2019), differentiating Equation (13) m-times with for the enclosed parameter $q$, and equating $q = 0$, and finally subdivide them by $m!$ we have the $m$th-order equation as

$$\ell_k[x_{km}(t) - \chi_m x_{m-1}(t)] = \hbar_k \Re_{km}\left(\vec{x}_{k(m-1)}(t)\right) \tag{18},$$

where

$$\Re_{km}(\vec{x}_{k(m-1)}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [T^\beta \phi_k(t,q) - g_k(t,\varrho,\phi_1(t,q),\phi_2(t,q)\phi_n(t,q))]$$
$$k = 1,2,\ldots n \tag{19},$$

with

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1. \end{cases}$$

Multiply both sides of Equation (18) by $\ell_k^{-1}$ and choose

$$\ell_k^{-1} = I^\beta \quad k = 1,2..n.$$

We have

$$x_{km}(t) = \chi_m x_{k(m-1)}(t) + \hbar_k I^\beta [\Re_{km}(\vec{x}_{k(m-1)}(t)] \tag{20}.$$

The $m$th-order Equation (18), which is linear, can be determined by any symbolic computation Maple software. From Equation (20), we get a power series solution

$$x_k(t) = \sum_{m=o}^{\infty} x_{km}(t) \quad 1 \le k \le n \tag{21}.$$

When $m \to \infty$ in Equation (21), we can get an accurate AAS of Eq. (3.4). The traditional approach to get interval values for convergence-control parameters $\hbar_k$ Is to plot the $\hbar_k$-Curve by drawing a curve of a certain proportion versus $\hbar_k$. Afterward, it was found that the traditional approach could not yield optimal values as declared by (Liao, 2010), which made (Odibat, 2020)

suggesting an optimization method for the optimal value of CCP by using squared residual error. Power series solution Equation (21) relies upon the convergence-control parameter vector.

The squared residual error in the *m*th-order of AAS is given as

$$\Delta(\hbar_1, \hbar_2) = \sum_{j=1}^{2} \int_{\Omega} (N_j[\sum_{k=0}^{m} x_1, \sum_{k=0}^{m} x_2])^2 d\Omega. \quad (22).$$

denotes the square residual error of the governing equation Equation (2) at the *m*th-order of approximation, where *m*=1, 2, and so on. At the first-order of an estimate, the squared residual error $\Delta_k$ are depending on $\hbar_k$ and the optimal values for $\hbar_k$ are obtained by solving non-linear algebraic equations

$$\frac{\partial \Delta(\hbar_k)}{\partial \hbar_k} = 0 \qquad k = 1, 2, \ldots . n. \quad (23).$$

## 3.1 The Convergence Analysis

**Theorem 1**. As long as the series $x_k(t) = x_{k0}(t) + \sum_{m=1}^{\infty} x_{km}(t)$ for $k = 1, 2, 3 \ldots . n$ converges where $x_{km}(t)$ is governed by Equation (18) under the definition Equation (19), it must be the result of Equation (11).

**Proof:** If we consider $\sum_{m=1}^{\infty} x_{km}(t)$ for $k = 1,2,3 \ldots . n$ converges to $x_k(t)$ then

$$\lim_{m \to \infty} x_{km}(t) = 0 \ \forall \quad k = 1, 2, \ldots . n \quad (24).$$

We can write

$$\sum_{m=1}^{\infty} \hbar_k \Re_{km}\left(\tilde{x}_{k(m-1)}(t)\right) = \sum_{m=1}^{\infty} \ell_k [x_{km}(t) - \chi_m x_{m-1}(t)], \quad (25)$$

$$= \lim_{n \to \infty} \sum_{m=1}^{n} \ell_k [x_{km}(t) - \chi_m x_{m-1}(t)], \quad (26)$$

$$= \ell_k x_{11}(t) + \left(\ell_k x_{22}(t) - \ell_k x_{21}(t)\right) + \left(\ell_k x_{nn}(t) - \ell_k x_{n(n-1)}(t)\right), \quad (27)$$

$$= \ell_k \left[\lim_{n \to \infty} \sum_{m=1}^{n} x_{nn}(t)\right], \quad (28)$$

$$= \ell_k \left[\lim_{n \to \infty} x_{nn}(t)\right] = 0 \quad \forall \quad k = 1, 2, \ldots . n. \quad (29).$$

Since $\hbar_k \neq 0$,

$$\Re_{km}\left(\tilde{x}_{k(m-1)}(t)\right) = 0. \quad (30)$$

Hence, using the above

$$\sum_{m=1}^{\infty} \Re_{km}(\tilde{x}_{k(m-1)}(t)) = \sum_{m=1}^{\infty} [T^{\beta} x_{k(m-1)} - g_k\left(t, \varrho, x_{1(m-1)}, x_{2(m-1)} \ldots . . x_{n(m-1)}\right)]$$

$$= \sum_{m=1}^{\infty} T^{\beta} x_{k(m-1)} - \sum_{m=1}^{\infty} g_k\left(t, \varrho, x_{k(m-1)}\right) \quad (32)$$

$$= T^{\beta} x_k t - g_k\left(t, \varrho, x_k(t)\right) \quad (33).$$

From Equations (30) and (33), we have

$$T^{\beta}x_k(t) = g_k\big(t, \varrho, x_k(t)\big) \quad \forall \quad k = 1, 2, \ldots . n \tag{36}$$

# 4 Numerical Examples and Results

**Example 1:** We consider test problem from (Schittkowski, 2012) [No 216]

$$minimize \quad g(x) = 100(x_1^2 - x_2)^2 + (x_1 - 1)^2 \tag{37},$$

$$subject\ to \quad h(x) = x_1(x_1 - 4) - 2x_2 + 12 = 0 \tag{38},$$

$$x_0 = (0, 0) \tag{39},$$

whose analytical solution is unknown, but the intended optimal solution is $x_1^* = 1.9993, x_2^* = 3.9998$. From Equation (5), we have

$$G(x, \varrho) = 100(x_1^2 - x_2)^2 + (x_1 - 1)^2 + \frac{1}{2}\varrho(x_1(x_1 - 4) - 2x_2 + 12)^2 \tag{40}.$$

Similarly from Equation (6), for $\varrho = 200$ gives

$$T^{\beta}x_1(t) = -400(x_1^2 - x_2)x_1 - 2(x_1 - 1) - \varrho(2x_1 - 4)(x_1^2 - 4x_1 - 2x_2 + 12) \tag{41},$$

$$T^{\beta}x_2(t) = 200(x_1^2 - x_2) + 2\varrho(x_1^2 - 4x_1 - 2x_2 + 12) \ \ 0 < \beta \leq 1, \tag{42},$$

$$x_1(0) = 0, \ \ x_2(0) = 0 \tag{43}.$$

Chosen a linear operator of the form

$$\ell_k^{-1} = I^{\beta}, \ k = 1, 2 \ldots n,$$

According to the formula Equation (19), we have

$$\begin{aligned}\Re_{1m}(\tilde{x}_{1(m-1)}(t), \tilde{x}_{2(m-1)}(t)) = T^{\beta}x_{1_{[m-1]}} - 2000\sum_{k=0}^{m-1} x_{1_{[m-1-\mathrm{k}]}}\sum_{j=0}^{n} x_{1_{[\mathrm{k}-\mathrm{j}]}}x_{1_{[\mathrm{j}]}} - \\ 9600\sum_{k=0}^{m-1} x_{1_{[m-1-\mathrm{k}]}}x_{1_{[\mathrm{K}]}} + 32002x_{1_{[m-1]}} + 3600x_{1_{[m-1]}}x_{2_{[m-1]}} + 6400x_{2_{[m-1]}} - \\ 38402(1 - \chi_m),\end{aligned} \tag{44},$$

$$\begin{aligned}\Re_{2m}(\tilde{x}_{1(m-1)}(t), \tilde{x}_{2(m-1)}(t)(t) = T^{\beta}x_{2_{[m-1]}} - 1800\sum_{k=0}^{m-1} x_{2_{[m-1-\mathrm{k}]}}x_{2_{[\mathrm{k}]}} - 3200x_{1_{[m-1]}} - \\ 1800x_{2_{[m-1]}} + 9600(1 - \chi_m),\end{aligned} \tag{45}$$

and the $m$th –order for $m \geq 1$ become

$$x_{1_m}(t) = \chi_m x_{1_{[m-1]}}(t) + \hbar_1 I^{\beta}[\Re_{1m}(\overleftarrow{x_{1_{[m-1]}}}(t))] \tag{46},$$

$$x_{2_m}(t) = \chi_m x_{2_{[m-1]}}(t) + \hbar_2 I^{\beta}[\Re_{2m}(\overleftarrow{x_{2_{[m-1]}}}(t))] \tag{47}.$$

From Equation (46) and Equation (47) above, we generate symbolic series solutions for Equation (41) and Equation (42) as:

$$x_1(1) = 38402\hbar_1 t, \tag{48}$$

$$x_2(1) = 9600\hbar_2 t, \tag{49}$$

$$x_1(2) = 4.423910400 \times 10^1 1\hbar_2 t^3 \hbar_1^2 + \hbar_1(-6.144704020 \times 10^8 \hbar_1 + (3.0720000 \times 10^7 \hbar_2)t^2 + 38402\hbar_1 t + \cdots \quad (50)$$

$$x_2(2) = 9600\hbar_2 t + 8727.272727\hbar_2^2 t^{\frac{11}{10}} - 9.6005000 \times 10^7 \hbar_2 t^2 \hbar_1 \quad (51)$$

From Equation (22), the "optimal values" of $(\hbar_1, \hbar_2)$ is estimated by

$$\frac{\partial \Delta(\hbar_1,\hbar_2)}{\partial \hbar_1} = 0, \ \frac{\partial \Delta(\hbar_1,\hbar_2)}{\partial \hbar_2} = 0 \quad (52).$$

It is found that

$$\Delta_1(1) = 1.636236158 \times 10^{16} - 7.612634534 \times^{20} \hbar_1 - 2.171187483 \times^{25} \hbar_1^2 - 2.004277445 \times 10^{28}\hbar_1^3 - 8.154151650 \times 10^{19}\hbar_2 + 1.409110792 \times 10^{24}\hbar_1\hbar_2 \quad (53)$$

$$\Delta_2(1) = 1.414477488 \times 10^{15} + 2.038537912 \times 10^{19}\hbar_2 \quad (54)$$

Our calculations indicated that, $\Delta_1(3)$ and $\Delta_2(3)$ has its minimum values at

$$\Delta_1(3) = -2.149 \times^{17} \text{ and } \Delta_2(3) = -2.678 \times^{11}.$$

and we have $\hbar_1 = 0.1041094886$, $\hbar_2 = -0.6938686201$ that gives an optimal solution to the problem

**Example 2**: We consider test problem from (Schittkowski, 2012) [No 1]

$$minimize\ g(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \quad (55)$$

$$subject\ \ to\ \psi(x) = 1.5 \leq x_2 \quad (56)$$

$$x_0 = (-2, 1) \quad (57)$$

whose analytical solution is unknown, but the intended optimal solution is $x_1^* = 1$, $x_2^* = 1$. From Equation (5), we have

$$G(x, \varrho) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + \varrho(max\{0, -1.5 - x_2\}^2) \quad (58)$$

Taking the differentiation of Equation (58), we have

$$T^\beta x_1(t) = -400x_1^3 + 400x_1x_2 - 2x_1 + 2, \quad (59)$$

$$T^\beta x_2(t) = 200x_1^2 - 200x_2, \quad (60)$$

$$x_1(0) = 0, \ x_2(0) = 0. \quad (61)$$

Following the same procedure above, we have

$$x_1(1) = -794\hbar_1 t, \quad (62)$$

$$x_2(1) = 600\hbar_2 t, \quad (63)$$

$$x_1(2) = -8.069580800 \times 10^{12}\hbar_1^4\hbar_2 t^5 + 1.260872000 \times 10^8 \hbar_1^4 t^4 - 1.479028739 \times 10^8 t^{\frac{31}{10}}\hbar_1^4 + \ldots \quad (64)$$

$$x_2(2) = (-60000\hbar_2^2 + 317600\hbar_1\hbar_2)t^2 + 545.4545455\hbar_2^2 t^{\frac{11}{10}} + 600\hbar_2 t) \quad (65)$$

and so on.

Our estimation showed that $\Delta_1(3)$ and $\Delta_2(3)$ has its minimum results at

$$\Delta_1(3) = -0.149 \times 10^{17}, \Delta_2(3) = 0.124 \times 10^{11}.$$

The optimal value of $\hbar_1, \hbar_2$ Are determine as $\hbar_1 = -0.05346$ and $\hbar_2 = 0.07546$.

**Table 1:** The relation and absolute error among (OHAM, $\beta = 1$) and (RK4, $\beta = 1$) solutions, of **example 1**.

| $t_k$ | OHAM$x_1$(t) | OHAM$x_2$(t) | RK4$x_1$(t) | RK4$x_2$(t) | Abs error $x_1$(t) | Abs error$x_2$ (t) |
|---|---|---|---|---|---|---|
| 0.0000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 0.0005 | 1.970111 | 3.871642 | 1.970899 | 3.871887 | 0.000788 | 0.000245 |
| 0.0010 | 1.977113 | 3.907789 | 1.978274 | 3.907993 | 0.001161 | 0.000204 |
| 0.0015 | 1.981099 | 3.922324 | 1.981384 | 3.922554 | 0.000285 | 0.00023 |
| 0.0018 | 1.982155 | 3.930478 | 1.983132 | 3.930578 | 0.000977 | 0.00010 |
| 0.0020 | 1.993105 | 3.995556 | 1.994252 | 3.995654 | 0.001147 | 0.000098 |

Tables 1 and 2 show the analytical approximate and numerical values, for **example1**, and **example 2** at a different time (t). It can be understood that the analytical approximate and numerical values are in close correspondence with a deficient absolute error, which makes OHAM an excellent method to solve non-linear FOCOP.

**Table 2:** The relation and absolute error among (OHAM, $\beta = 1$) and (RK4, $\beta = 1$) solutions of **example 2.**

| $t_k$ | OHAM$x_1$(t) | OHAM$x_2$(t) | RK4$x_1$(t) | RK4$x_2$(t) | Abs error $x_1$(t) | Abs error$x_2$ (t) |
|---|---|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 2.0 | 0.156230 | 0.023457 | 0.161481 | 0.023694 | 0.005251 | 0.000237 |
| 4.0 | 0.691769 | 0.578784 | 0.693706 | 0.579773 | 0.001937 | 0.000989 |
| 6.0 | 0.830467 | 0.788985 | 0.830548 | 0.789065 | 8.1E-05 | 8E-05 |
| 8.0 | 0.917518 | 0.970234 | 0.927053 | 0.970426 | 0.009535 | 0.000192 |
| 10.0 | 0.997398 | 0.987894 | 0.998826 | 0.993629 | 0.001428 | 0.005735 |

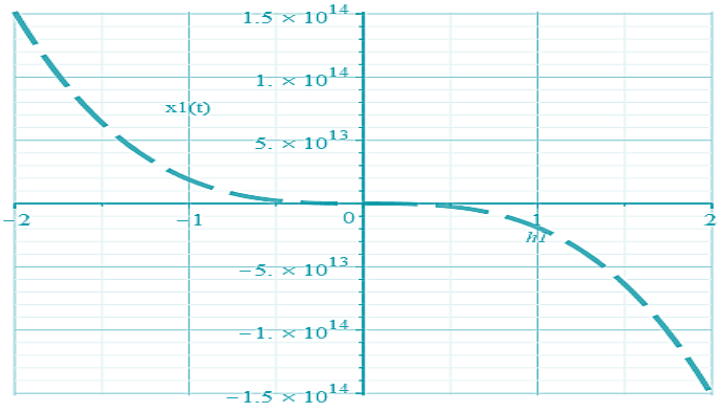


**Figure 1.** The h1-curve of OHAM for example 1 at (t=0.005) of $x_1(t)$.

The h1-curve in Figure 1 shows the convergence domain of the solution of **example 1**, between an interval of $-0.5 < \hbar_1 < 0.5$ at $x_1(t)$ and provides a clear region of the optimal values for the convergence of the proposed solution method. Figure 2 shows the domains of convergence of the series solution given by h2 at x2(t) of example 1 and provided a clear region of the optimal values for the convergence of the proposed solution method.

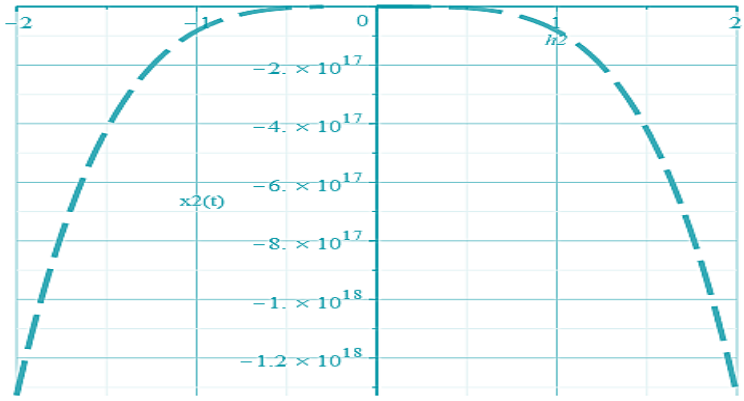


**Figure 2:** The h2-curve of OHAM for example 1 at (t=0.005) of x2 (t).

Figures 3 and 4 shows mathematical simulations of the proposed method at x1(t) and x2(t) of example 1, respectively. The mathematical simulation illustrates OHAM solution as an excellent tool to solve the problem.

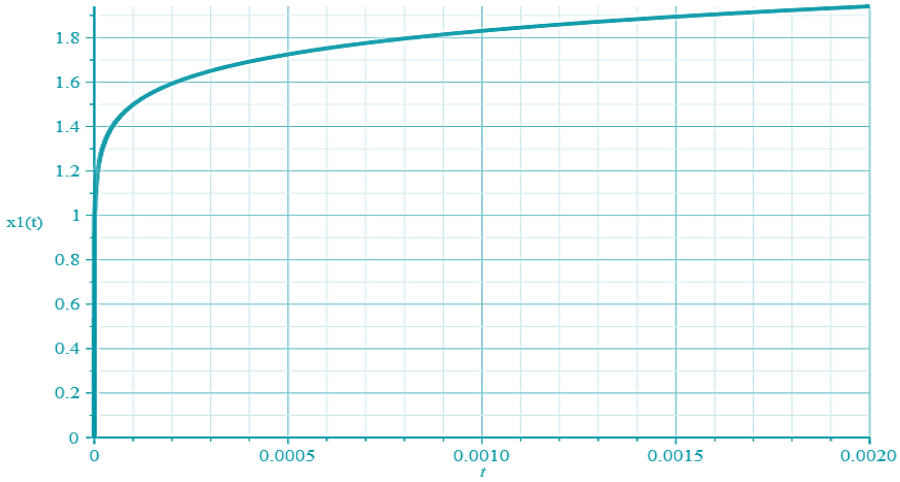


**Figure 3.** The mathematical simulation of OHAM solution for $\beta$=0.9 of **example 1** at the third-order approximation of x1(t).

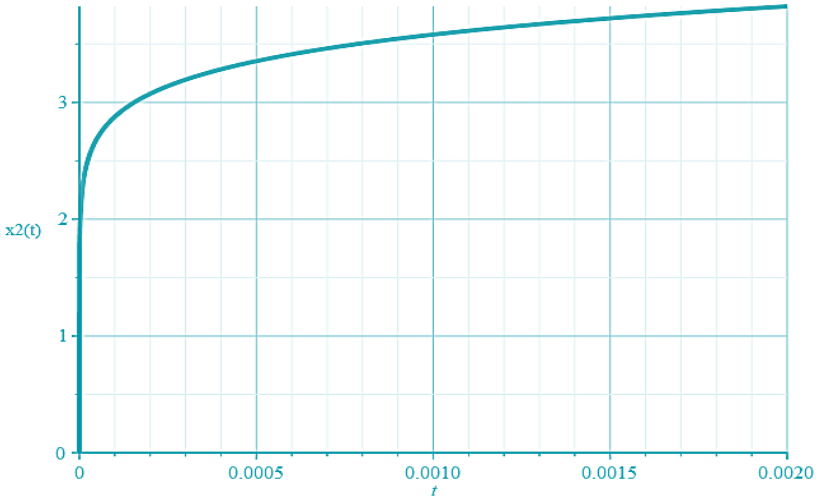


**Figure 4.** The mathematical simulation of OHAM solution for $\beta$=0. For **example 1** at the third-order approximation of x2 (t).

Figures 5 and 6 shows the comparisons between different values of OHAM and RK4 method for justification at $x1(t)$ and $x2(t)$ of example 1. The comparison shows that OHAM would perform fast convergence to the optimal solutions. From Figure 6, the solution from OHAM is very precise with the expected optimal solution as the values of ($\beta$=1) approaches 1. The h1-curve in Figure 7 shows the domain of convergence of the series solution of example 2, between an interval of -1<$\hbar$_1<1 at x1 (t).

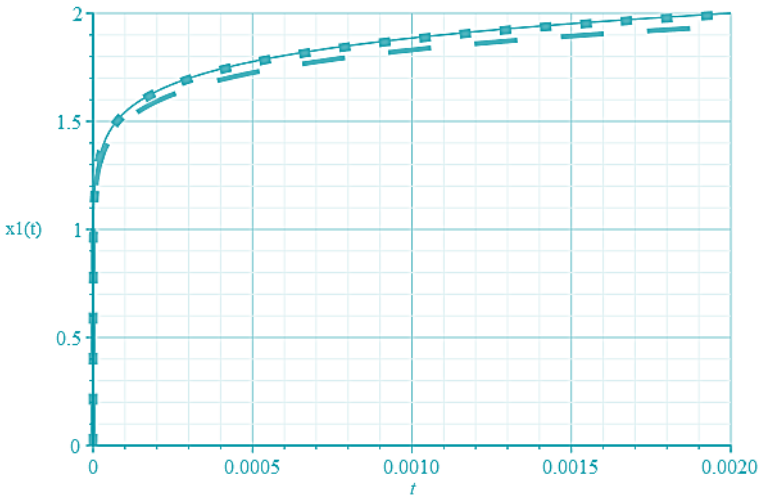


**Figure 5.** Comparisons between OHAM ($\beta = 0.9, space\ dash, \beta = 1, dot$) and RK4 ($\beta = 1, solid$) *at* $x1(t)$ of **example 1**. It is seen in figure 5 that the solution from OHAM is very precise with the expected optimal solution as the values of ($\beta$=1) approaches 1.

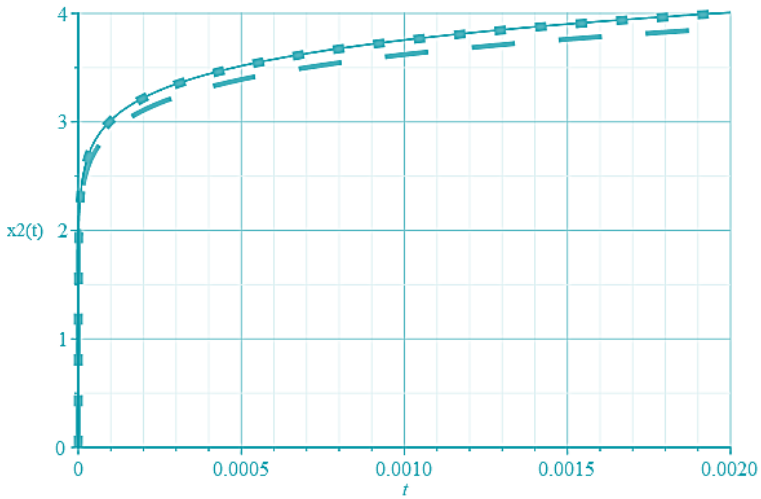


**Figure 6.** Comparisons between OHAM ($\beta = 0.9, space\ dash, \beta = 1, dot$) and RK4 ($\beta = 1, solid$) *at* $x2(t)$ of **example 1**.

Figure 7 shows the domains of convergence of the series solution given by $\hbar_1$ at x1(t) and $\hbar_2$ at x2(t) of example 2 and provided a clear region of the optimal values for the convergence of the proposed solution method.

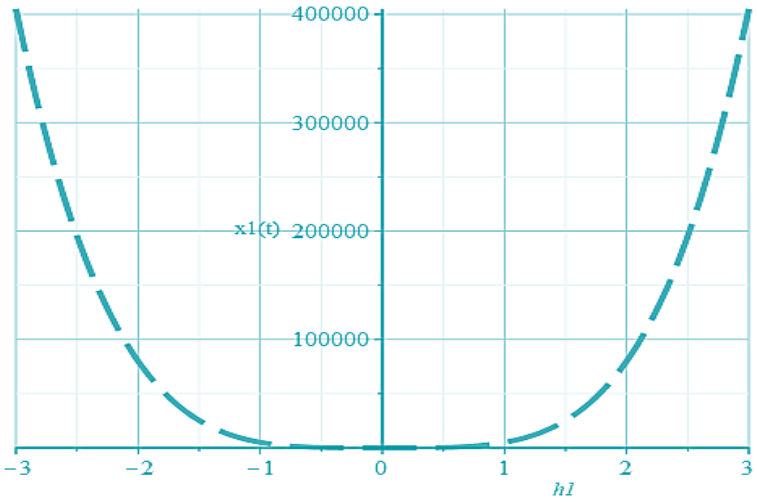


**Figure 7.** The h1-curve of OHAM, for example 2 at (t=2) of x1 (t).

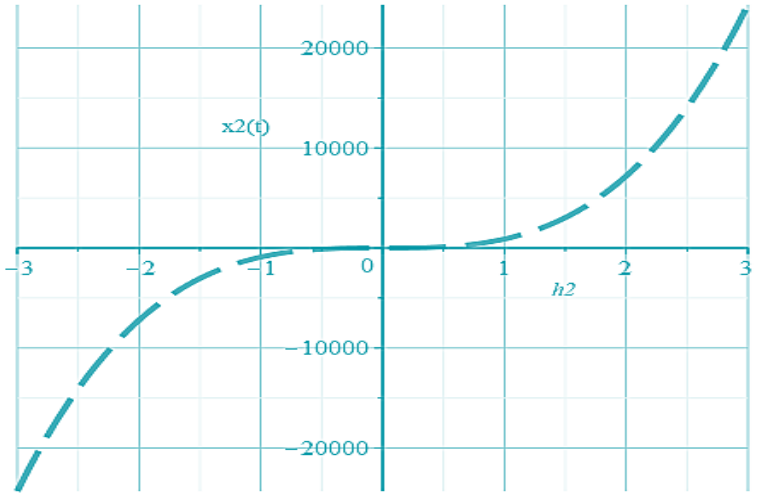


**Figure 8.** The h2-curve of OHAM, for example 2 at (t=2) of x2 (t). The h2-curve in figure 8 shows the domain of convergence of the series solution of example 2, between an interval of $-1 < \hbar_2 < 1$ at x2 (t).

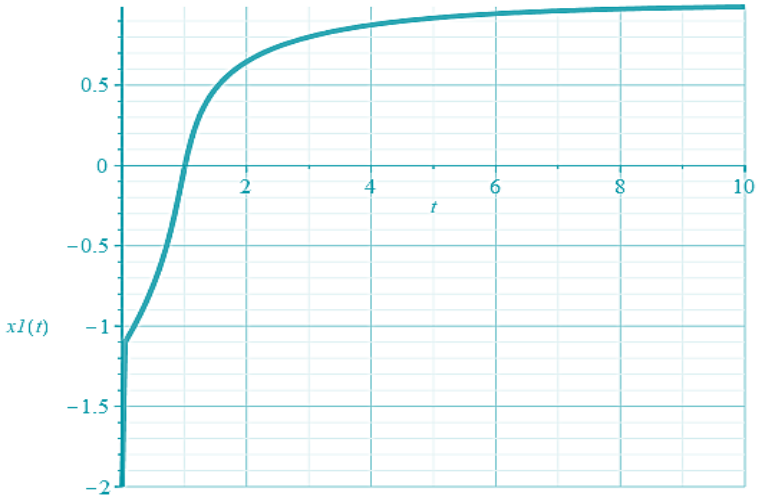


**Figure 9.** The mathematical simulation of OHAM solution for $\beta$=0.
For **example 2** at the third-order approximation of x1(t).

Figures 9 and 10 show mathematical simulations of the proposed method at x1(t) and x2(t) of example 2. The mathematical simulation illustrates OHAM solution as an excellent tool to solve the problem.

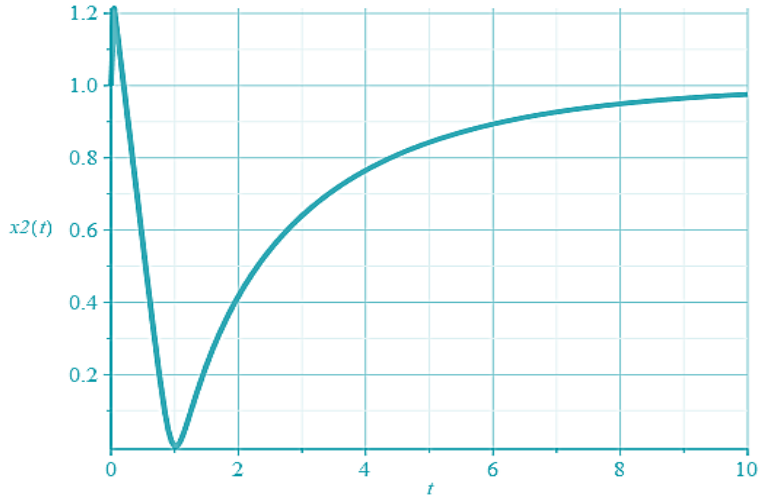


**Figure 10.** The mathematical simulation of OHAM solution for $\beta$=0. For **example 2** at the third-order approximation of x2(t).

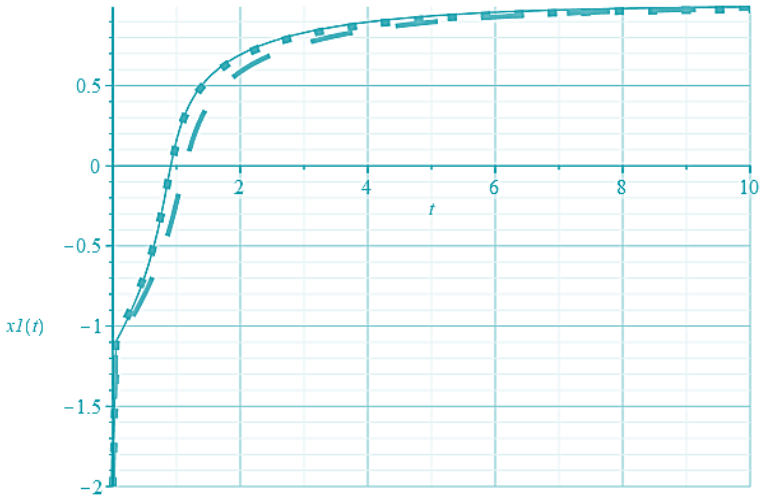


**Figure 11.** Comparisons between OHAM ($\beta = 0.9, space\ dash, \beta = 1, dot$) and RK4 ($\beta = 1, solid$) $at\ x1(t)$ of **example 2**.

It is seen in Figure 11 that the solution from OHAM is very precise with the expected optimal solution as the values of ($\beta$=1) approaches 1. From Figure 12, the solution from OHAM is very precise with the expected optimal solution as the values of ($\beta$=1) approaches 1

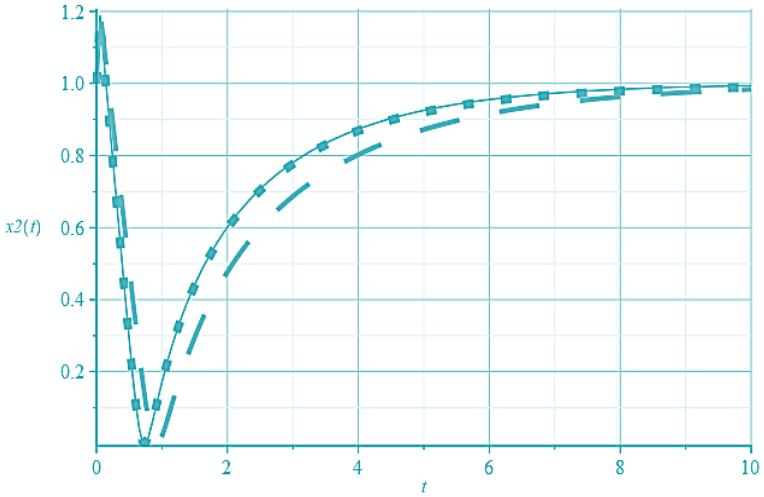


**Figure 12.** Comparisons between OHAM ($\beta = 0.9, space\ dash, \beta = 1, dot$) and RK4 ($\beta = 1, solid$) $at\ x2(t)$ of **example 2**.

## 5 Discussion

The results are given by $(\hbar_1, \hbar_2)$ Using OHAM shows the strength of the method on the convergence of the series solution through h-curves plots of the systems for ($\beta$=0.9, 1.) We showed that even three-order terms of the analytical approximation of the solutions are enough to get an accurate solution. It is understandably that the exactness can be optimized by computing a few more terms of the approximate analytical solutions.

## 6 Conclusion

This paper solved non-linear FOCOP by OHAM after the third order of approximation solution was used with fast convergence and accurate solutions. The numerical comparison among the RK4 and OHAM ($\beta$=1), shows that OHAM would perform fast convergence to the optimal solutions as ($\beta$=1) tends to 1, the integer-order solution for the system is reclaimed OHAM is an effective tool for obtaining an AAS for non-linear FOCOP.

## 7 Availability of Data and Material

Information can be made available by contacting the corresponding author.

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