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# ON REALIZATION OF LIMIT POLYGONS IN SEQUENTIAL PROJECTION METHOD

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**ABSTRACT**

In this article, we study the properties of the algorithm for solving systems of linear equations based on the sequential projections of an initial approximation point on the hyperplanes, defined by the equations of the system (Kaczmarz algorithm). We consider the case of overdetermined systems when the sequence of approximations converges to a limit cycle, the points of which we regard as the vertices of a limit polygon. Although the proof of convergence to the limit polygon is known, we are discussing a simplified version, relating to the case of general position and clarifying the main idea of the proof. The properties of the limit polygon are little studied, but at the same time, they are important for applications. We explain that, with a proper choice of a system of equations, the limit polygon can be any predefined polygon. In other words, there are no restrictions on the type of limit polygon.



## 1. INTRODUCTION

The sequential projection algorithm for solving systems of linear equations was first described in Kaczmarz's work [6] (paper [6] is a translation of the original Kaczmarz's paper published in 1937). This algorithm has a simple and visual geometric meaning and later it was repeatedly rediscovered. For some mysterious reasons, the algorithm was very rarely mentioned in textbooks and manuals on computational linear algebra, at least until recently, when the algorithm was intensively used in computed tomography problems. At the same time, the algorithm is very popular in the scientific literature and there are a large number of sources devoted to the description of the algorithm, its generalizations and the range of related tasks and applications. In particular, the algorithm and its applications are described in the excellent book [2] and there is also an extensive bibliography. The issues of convergence of the algorithm in the case of arbitrary rectangular systems of linear equations were studied, for example, in [11] for the original version, in [1], [5], [7] for the block version, in [3],

[11] for versions with relaxation, in [4], [8], [9], [10] for the version with the randomized choice of the hyperplanes sequence. This listing of articles, of course, does not pretend to be complete, and for a more detailed bibliography, we can recommend, as already mentioned, the book [2]. In connection with the universality of the algorithm, the case of overdetermined systems is especially interesting, when the modified algorithm allows finding pseudo-solutions and can be considered as an alternative for pseudo-inversion.

## 2. METHOD DESCRIPTION

From the geometric point of view, the Kaczmarz algorithm for solving a system of $k$ linear equations with $n$ unknowns can be described as a sequential cyclic projection of an arbitrary initial point on the hyperplanes in $\mathbf{R}^n$, determined by the equations of the system. Here it is necessary to distinguish two cycles: the "internal" cycle of the length $k$ according to all equations of the system and the "external" cycle corresponding to the multiple uses of the internal cycle to achieve required accuracy. The algorithm converges to a solution for consistent systems. In the case of inconsistent systems, the algorithm (without the relaxation multiplier) converges cyclically to some sequence of points on the hyperplanes. Such a sequence can be interpreted as a sequence of polygon vertices in space. We call this polygon the limit polygon (or limit cycle) of the algorithm. Here are two immediate questions related to the concept of a limit polygon: a description of the class of limit polygons (which polygons in space may be limit for the Kaczmarz algorithm); a description of the connection of limit polygons with pseudo-solutions of systems, for example, does a pseudo-solution (or a geometric pseudo-solution, see the definition in [2]) belong to the convex hull of a limit polygon. In this article, we give a simple proof of the known fact that the algorithm converges to a limit polygon and show that there are no restrictions on a limit polygon, that is, any preselected polygon is a limit polygon for some overdetermined system of linear equations. Note that there is a version of the so-called double Kaczmarz algorithm (DART - see [2]), which allows to exclude work with inconsistent systems and find the pseudo-solutions as the only solution to the some modified overdetermined (but already consistent) system of equations. However, the study of the properties of a conventional algorithm in the case of inconsistent systems remains relevant.

## 3. PROOF OF CONVERGENCE

Let $AX = B$ be a system of $k$ linear equations with $n$ unknowns. Denote by $L_1, L_2, \cdots, L_k$ the hyperplanes in $\mathbf{R}^n$, defined by the equations of the homogeneous system $AX = 0$. Affine hyperplanes $\widetilde{L_1}, \widetilde{L_2}, \cdots \widetilde{L_k}$, determined by the equations of the original system $AX = B$, are derived from hyperplanes $L_1, L_2, \cdots, L_k$ by shifts on normal vectors $n_{\perp i} \perp L_{\perp i}$,

$$\widetilde{L_\iota} = \bar{n}_i + L_i \qquad (1).$$

Let $P_i$ denote the linear projection operator $\mathbf{R}^n \to L_i$ , then the affine projection operator $\tilde{P}_i : \mathbf{R}^n \to \widetilde{L_\iota}$ can be written as

$$X \to \tilde{P}_i\,(X) = P_i(X) + \bar{n}_i \in \widetilde{L_\iota} \qquad (2).$$

In these notations, the result of one internal cycle of the algorithm of sequential projections (without the random choice of the sequence of hyperplanes) can be written as

$$X_s \to X_{s+1} = \tilde{P}_k \circ \tilde{P}_{k-1} \dots \circ \tilde{P}_2 \circ \tilde{P}_1 (X_s) \in \widetilde{L_k} \quad (3),$$

or, taking (2) into account as

$$X_s \to X_{s+1} = [\![P_k( \dots (P]\!]_2(P_1(X_s) + \bar{n}_1) + \bar{n}_2) \dots ) + \bar{n}_k \in \widetilde{L_k} \quad (4).$$

Thus, the final formula for the result of one internal cycle is

$$X_s \to X_{s+1} = P_k \circ P_{k-1} \dots \circ P_2 \circ P_1 (X_s) + N \in \widetilde{L_k} \quad (5).$$

where $N$ is a "universal" vector dependent only on the sequence $P_1 \dots P_k$ and $\bar{n}_1 \dots \bar{n}_k$ i.e. does not depend on $X_s$). Denote the composition $P_k \circ P_{k-1} \dots \circ P_2 \circ P_1$ by $P$, then formula (5) can be written as

$$X_s \to X_{s+1} = P(X_s) + N \quad (6).$$

The iterations of the internal cycle, i.e. "external" cycle, are described by the formulas

$$X_0 \to P(X_0) + \bar{N} = X_1 \to P(P((X_0) + N) + N = X_2 \to \square \quad (7),$$

where

$$X_0 - \forall,$$

$$X_1 = P(X_0) + N \in \widetilde{L_k}$$

$$X_2 = P^2(X_0) + P(N) + N \in \widetilde{L_k}$$

$$X_{s+1} = P^{s+1}(X_0) + P^s(N) + \cdots P(N) + N \in \widetilde{L_k}$$

$$\dots$$

Note that the norm $\|P\| < 1$ if the subspaces $L_1, L_2, \cdots, L_k$ have no common vectors, except for $\mathbb{0}$ (and this is the general position case). Indeed, the projection operator $P_i$ is fixed on $L_i$ and reduces the lengths of all other vectors. Therefore, the condition $|P(X)| = |X|$ means that $X \in L_i$ for all $i$ but it is forbidden by condition. Thus, $\|P(X)\| \le q|X|$ for some $q \in (0; 1)$ and hence the first term in $X_{s+1} = P^{s+1}(X_0) + P^s(N) + \cdots P(N) + N \in \widetilde{L_k}$ goes to zero when $s \to \infty$. The rest is estimated as $\|P^s(N) + \cdots P(N) + N\| \le \|P^s(N)\| + \cdots + \|P(N)\| + \|N\| \le |N|(q^s + \cdots + q + 1)$, and therefore converges. We proved that the Kaczmarz algorithm converges for any starting point $X_0$, while the limit point depends on the numbering of the equations and belongs to the affine hyperplane $\widetilde{L_k}$, which is determined by the last equation of the system. The result also implies that every sequence $X_{i+s} \in \widetilde{L_\iota}$ for every fixed $i = 1,2, \dots, k$ converges to a point on the affine hyperplane $\widetilde{L_\iota}$. Therefore, for overdetermined systems the sequence of approximations in the algorithm converges to a limit cycle, the points of which we regard as the vertices of a limit polygon.

Let us also explain the convergence of the Kaczmarz algorithm in the case when the system is consistent and has a unique solution. First of all, note that the affine shift does not affect the geometry of the Kaczmarz algorithm. This observation makes the proof of convergence almost obvious. Let $X_{sol}$ be the unique solution. We consider a new system, obtained from the initial one by a shift on the vector $X_{sol}$. Then the initial system $AX = B$ with the unique solution $X_{sol}$ turns into the system $AX = 0$ with the unique solution at point 0. In this case, $L_i = \tilde{L}_i$ and $\bar{n}_i = 0\ \forall i$, therefore, $N = 0$. The formula for iterations is reduced to $X \to P(X)$, where $\|P\| < 1$, which implies the convergence.

## 4. PREDEFINED LIMIT POLYGONS

The following construction shows that in the space of any dimension there are no restrictions on the type of a limit polygon: any given polygon is realized as a limit for some overdetermined system of linear equations. Let $M_1, M_2, \dots, M_k$ be an ordered set of points in the space $\mathbf{R}^n$, $k > n$ the first equation of the system is the equation of the hyperplane passing through the point $M_1$ perpendicular to the vector $\overline{M_k M_1} = N_1$. The second equation of the system is the equation of the hyperplane passing through the point $M_2$ perpendicular to the vector $\overline{M_1 M_2} = N_2$ and so on; the last equation of the system is the equation of the hyperplane passing through the point $M_k$ perpendicular to the vector $\overline{M_{k-1} M_k} = N_k$. It is clear that the polygon $M_1, M_2, \dots, M_k$ will be the cycle of the algorithm for the initial point $X_0 = M_k$, and since it was proved above that the limit cycle is unique, this polygon will be the limit cycle of the algorithm for the constructed system.

## 5. CONCLUSION

The sequential projection algorithm is an efficient iterative algorithm for solving systems of linear equations. This algorithm is distinguished by its universality in terms of the type of system and, if properly interpreted, it can be used for many systems, including overdetermined and inconsistent. In the last case an important task is the study the properties of the limit polygons. In this article, we showed that any predefined polygon is a limit polygon for some overdetermined system of equations.

## 6. MATERIAL AND DATA AVAILABILITY

Information regarding this study is available from the corresponding author.

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