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SOLVING FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD USED IN OPTIMAL CONTROL THEORY

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ABSTRACT

In this paper, the Adomian decomposition method (ADM) is used to solve the fractional differential equations under Caputo derivative. This study reviews basic definitions of fuzzy set and fractional calculus. This work studies and discusses Adomian decomposition method under Fuzzy fractional Caputo derivative. Since the ADM approximates the exact solution as an infinite series, then the convergence theorem is considered with successive iterations. For illustration, an example is given to compare between exact and approximate solutions. The uniformly convergence of sequence $\{y_i(t)\}$ with various types of differentiability to the exact solution is proved.

1. INTRODUCTION

Fractional calculus is a benefit branch in mathematics, chemistry, optics, control theory, engineering and etc that have studied in recent years. Fractional calculus was slowly prepared at first, but it has been a powerful implement in various applications. Fuzzy fractional differential equations is casted by Agarwal et al. (2010). They have drafted the Riemann-Liouville differentiability concept under the Hukuhara differentiability to solve fuzzy fractional differential equations. In Arshad & Lupulescu (2011) and Allahviranloo (2005), the existence and uniqueness are proved for solutions of fuzzy fractional differential equations under Riemann-Liouville differentiability. Allahviranloo et al. (2014) proved the existence and uniqueness results for fuzzy fractional integral and Integra-differential equations involving Riemann-Liouville differential operators. The famous used method in solving fractional differential equations is the Caputo fractional derivative. Based on generalized Hukuhara derivative (2001), the opinion of fractional Caputo derivative is introduced in Armand & Gouyandeh (2013), afterwards fuzzy fractional differential equations are investigated under this kind of

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differentiability. Allahviranloo et al. in 2009 proved the existence and uniqueness solution of fuzzy fractional differential equations (FFDEs) under Caputo's gH-differentiability.

In the present paper, we attempt an analytical method to solve FFDEs. To this end, we adopted the Adomian decomposition method (ADM) to solve FFDEs. The Adomian decomposition method (ADM) can be used to solve various mathematical models or system of equations involving algebraic equations, differential equations, Integra-differential equations and linear or non-linear equations (Adomian 1988; Adomian, 1994; Abbaoui & Cherruault, 1994; Wazwaz, 2001; Wazwaz, 2005). In Wazwaz (2001) ADM has employed for solving non-linear fractional differential equations. Specifically, in Daftardar-Gejji & Jafari (2005) and Jafari & Daftardar-Gejji (2006) ADM has used for solving systems of fractional differential equations (linear and nonlinear). Convergence of Adomians method has been studied in (Abdelrazec & Pelinovsky, 2011; Abbaoui & Cherruault, 1994; Ghanbari & Nuraei, 2014; Hosseini & Nasabzadeh, 2006).

2. PRELIMINARIES

In this section we present some definitions and useful concepts of fractional calculus and fuzzy set. Let us consider $\Re_F$ be a set of all fuzzy numbers on $\Re$.

Definition 2.1. A fuzzy number is a map $u: \Re \to [0,1]$ which satisfies (Zimmermann, 1991)

i. u is upper semi-continuous.

ii. u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$for all $x, y \in \Re, \lambda \in [0,1]$.

iii. u is normal, i.e., $\exists x_0 \in \Re; u(x_0) = 1$.

iv. $supp\ u = \{x \in \Re | u(x) > 0\}$ is the support of the u, and its closer $(supp\ u)$ is compact.

The metric structure is given by the Hausdorff distance.

$D: \Re_F \times \Re_F \to \Re_+ \cup \{0\}$,

$D(u,v) = sup_{\alpha\in[0,1]} max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}$.

$(\Re_F, D)$ is a complete metric space and the following properties are well known: (Ghanbari & Nuraei, 2014).

i. $D(u \oplus w,\ v \oplus w) = D(u,v)\ \forall\ u,v,w \in \Re_F$

ii. $D(u \oplus w,\ \tilde{0}) = D(u, \tilde{0}) + D\ (v, \tilde{0}\ )\ \ \forall u,v \in \Re_F$

iii. $D(u \oplus w,\ w \oplus z) \leq D(u,w) +\ D(v,z)\ \ \forall\ u,v,w,z \in \Re_F$

iv. $D(\lambda.u\ , \lambda.v) = |\lambda| D(u,v)\ \forall\ u,v \in \Re_F\ \ \lambda \in \Re_F$

Definition 2.2. The generalized Hukuhara difference of two fuzzy numbers $u, v \in \Re_F$ (gH-difference for short) is defined as follows (Bede & Stefanini, 2013):

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v \oplus W, \\ (ii) & v = u \oplus (-1)w. \end{cases}$$

Definition 2.3. The generalized Hukuhara derivative of a fuzzy-valued function $f: (a,b) \to \Re_F$ at x_0 is defined as (Stefanini & Bede, 2009)

$$(f)'_{gH}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) \ominus_{gH} f(x_0)}{h}$$

Also, we say that

i. f is $[(i)-gH]$ differentiable at x_0 if

$(f)'_{gH}(x_0;\alpha) = [f'_-(x_0,\alpha),\ f'_+(x_0,\alpha)],\ 0 \le \alpha \le 1.$

ii. f is $[(ii)-gH]$ differentiable at x_0 if

- $(f)'_{gH}(x_0;\alpha) = [f'_{\mp}(x_0,\alpha),\ f'_-(x_0,\alpha)],\ 0 \le \alpha \le 1.$

Definition2.4. Let $P_k(\Re_F)$ denote the family of all nonempty, compact and convex subsets of $\Re^n$ and define the addition and scalar multiplication in $P_k(\Re^n)$ is usual (Armand & Gouyandeh, 2013). A mapping $f: I \to \Re^n$ is strongly measurable, if for all $\alpha \in [0,1]$ the set-valued mapping $f_n: I \to P_k(\Re^n)$ defined by

$f_\alpha(t) \to [f(\alpha)]^t$

Is Lebesgue measurable, when $P_k(\Re^n)$ is endowed with the topology generated by the Hausdorff metric D.

Definition 2.5. A mapping $f: I \to \Re^n$ is called integrably bounded if there exists an integrable function h such that $\|x\| \le h(t)$ for all $x \in f_0(t)$ (Armand & Gouyandeh, 2013).

Definition 2.6. A strongly measurable and integrably bounded mapping $f: I \to \Re^n$ is said to be integrable over I if $\int_t f(t)dt \in \Re_F$ (Armand & Gouyandeh, 2013).

Definition 2.7. Let $f: [a,b] \to \Re^n$. The fuzzy fractional Riemann-Liouville integral of fuzzy-valued function f is defined as follows (Abasbandy et al., 2012)

$$(J_a^v f)(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t)dt,$$

for $0 \le V \le 1$. For V=1, we set $I_a^l = I$, the identity operator.

Let us denote $C_F[a,b]$ as the space of all continuous fuzzy-valued functions on$[a,b]$. Also, we denote the space of all integrable fuzzy-valued functions on interval $[a,b]$ by $L_F[a,b]$Let $f_{gH}^{(n)} \in C_F[a,b] \cap L_F[a,b]$, thus the fuzzy gH -fractional Caputo differentiable of fuzzy-valued function f (CF $[gH]$-differentiable for short) is defined as following.

Definition 2.8. Consider $f: [a,b] \to \Re^n$. Fractional derivative of f under generalized Hukuhara differentiability in the Caputo sense is defined as (Abasbandy et al., 2012)

$$({}_{gH}D_*^v f)(x) = \frac{1}{\Gamma(n-v)} \int_a^x (x-t)^{n-v-1} f_{gH}^{(n)}(t)dt, \qquad n-i < v \le n,\quad n \in N, x > a$$

Also we say that f is ${}^{CF}[(i)-gH]$-differentiable at x_0 if

$$({}_{gH}D_*^v f)(x_0;\alpha) = [(D_*^v f_-)(x_0;\alpha),\ (D_*^v f_+)(x_0;\alpha)] \quad 0 \le \alpha \le 1 \tag{1}$$

And so f is ${}^{CF}[(ii)-gH]$-differentiable at x_0 if

$$({}_{gH}D_*^v f)(x_0;\alpha) = [(D_*^v f_+)(x_0;\alpha),\ (D_*^v f_-)(x_0;\alpha)] \quad 0 \le \alpha \le 1 \tag{2}$$

In this paper, we only consider ${}^{CF}[gH]$-differentiable of order $0 \le V \le 1$ for fuzzy-valued function f. Furthermore

$(D_*^v f_-)(x_0;\alpha)=\frac{1}{\Gamma(1-v)}\int_a^x \frac{(f_-)\prime(t,\alpha)dt}{(x-t)^v}$

And

$(D_*^v f_+)(x_0;\alpha)=\frac{1}{\Gamma(1-v)}\int_a^x \frac{(f_+)\prime(t,\alpha)dt}{(x-t)^v}$

So

$({}_{\text{gH}}D_*^v f)(x;\alpha) = \frac{1}{\Gamma(1-v)}[\min\left\{\int_a^x \frac{(f_-)\prime(t,\alpha)dt}{(x-t)^v}, \int_a^x \frac{(f_+)\prime(t,\alpha)dt}{(x-t)^v}\right\},\ max\left\{\int_a^x \frac{(f_-)\prime(t,\alpha)dt}{(x-t)^v}, \int_a^x \frac{(f_+)\prime(t,\alpha)dt}{(x-t)^v}\right\}]$

Lemma 2.1. Suppose that $f:[a,b]\to\Re_F$ be a fuzzy-valued function and $f'_{gH}\in C_F[a,b]\cap L_F[a,b]$ Then (Allahviranloo et al., 2013) $J_a^v(\text{gH}D_*^v f(x))(x) = f(x)\ominus gHf(a), 0\le\alpha\le 1$.

3. ADOMIAN DECOMPOSITION METHOD UNDER FUZZY FRACTIONAL CAPUTO DERIVATIVE

Consider a fuzzy Caputo fractional differential equation with fuzzy initial value as follows

$$\begin{cases} \left({}_{gH}D_*^v y(t)\right) = F(t,y(t)) = L_y(t) + N_y(t) \\ y(t_0) = y_0 \in \Re_F \end{cases} \tag{3}$$

So that $0\le V\le 1$ and $F:[a,b]\times\Re_F\to\Re_F$ is supposed to be continuous. Also ${}_{\text{gH}}D_*^v$ denote the generalized Hukuhara Caputo derivative of y(t) which must be determined. Also L is a linear operator and N represents the nonlinear operator.

The Adomian supposes that the unknown function y(t) can be written by a sum of components as

$y(t) = \sum_{i=0}^{\infty} y_i,$

And so, the nonlinear operator $Ny(t)$ is represented by an in finite series as

$$N(t) = A(t) = \sum_{i=0}^{\infty} A_i,$$

Such that $A_i(t)$ are called Adomian's polynomials, given as

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{j=0}^{\infty}\lambda^j y_j\right)\right]_{\lambda=0} \qquad i = 0,1,2,\ldots.$$

Finally to calculate terms of series $\sum_{j=0}^{\infty} y_i$, we use the iterated scheme that explain in theorem 3.1.

Lemma3.1. Let $y_i, i = 0,1,2,\ldots,n$ be fuzzy continuous functions. Thus $({}_{\text{gH}}D_*^v \sum_{i=0}^{n} y_i(t)) = \sum_{i=0}^{n} D_*^v y_i(t)$.

Proof: We prove this lemma by mathematical induction and (i)-gH-differentiability.

$$\begin{aligned}\left({}_{gH}D_*^v\left(y_0(t)+y_1(t)\right)\right) &= \left[D_*^v\left(\underline{y}_0(t)+\underline{y}_1(t)\right), D_*^v\left(\overline{y}_0(t)+\overline{y}_1(t)\right)\right] \\ &= \left[D_*^v\left(\underline{y}_0(t)\right)+D_*^v\left(\underline{y}_1(t)\right), D_*^v\left(\overline{y}_0(t)\right)+D_*^v\left(\overline{y}_1(t)\right)\right] \\ &= D_*^v\left[\left(\underline{y}_0(t)\right),\left(\overline{y}_0(t)\right)\right]+D_*^v\left[\left(\underline{y}_1(t)\right),\left(\overline{y}_1(t)\right)\right] \\ &= {}_{gH}D_*^v\left(y_0(t)\right) + {}_{gH}D_*^v\left(y_1(t)\right).\end{aligned}$$

Suppose that

$$\left({}_{gH}D_*^{v}\sum_{i=0}^{n-1} y_i(t)\right)=\sum_{i=0}^{n-1}{}_{gH}D_*^{v} y_i(t).$$

Thus

$$\left({}_{gH}D_*^{v}\sum_{i=0}^{n} y_i(t)\right)={}_{gH}D_*^{v}\left[\sum_{i=0}^{n-1} y_i(t)+y_n(t)\right]=\sum_{i=0}^{n}{}_{gH}D_*^{v} y_i(t).$$

Theorem 3.1. Assume that $y(t)=\sum_{i=0}^{\infty} y_i$ in Eq.(3.1), so that terms of series can be in the various types of differentiability. We consider four cases in this theorem.

Case (I). Let $y_i(\mathrm{t})$, $\forall \mathrm{i}\ (\mathrm{i}=0,1,\ldots)$ is (i)-gH-differentiable, then

$$y_0(t)=y_0(t_0)$$
$$y_i(t)=J_{t_0}^{v}L_{y_{i-1}}(t)\oplus J_{t_0}^{v}A_{i-1}(t).\quad i=1,2,\ldots \tag{4}$$

Case (II). Let $y_i(\mathrm{t})$, $\forall \mathrm{i}\ (\mathrm{i}=0,1,\ldots)$ is (ii)-gH-differentiable, then

$$y_0(t)=y_0(t_0)$$
$$y_i(t)=\tilde{0}\ominus(-1)J_{t_0}^{v}L_{y_{i-1}}(t)\ominus(-1)\,J_{t_0}^{v}A_{i-1}(t).\quad i=1,2,\ldots \tag{5}$$

Case (III). Let i is even, $y_i(t)$ is (i)-gH-differentiable and j is odd, $y_j(t)$ is (ii)-gH-differentiable, then

$$\begin{aligned} &y_0(t)=y_0(t_0),\\ &y_i(t)=\tilde{0}\oplus J_{t_0}^{v}Ly_{i-1}(t)\oplus J_{t_0}^{v}A_{i-1}(t), && i \text{ is even}\\ &y_j(t)=\tilde{0}\ominus(-1)J_{t_0}^{v}Ly_{j-1}(t)\ominus(-1)J_{t_0}^{v}A_{j-1}(t). && j \text{ is odd} \end{aligned} \tag{6}$$

Proof: Assume that $y_i(\mathrm{t})$ $i=0,1,2,\ldots$ is continues function.

(I) Let $y_i(\mathrm{t})$, $\forall \mathrm{i}\ (\mathrm{i}=0,1,\ldots)$ is differentiable as in Definition 2.3 (i).Thus

$$\left({}_{gH}D_*^{v}y(t)\right)=\left({}_{gH}D_*^{v}\sum_{i=0}^{\infty} y_i(t)\right)=F(t,\sum_{i=0}^{\infty} y_i(t))$$

By the Riemann-Liouville integral of two side of above equation we have

$$\begin{aligned} J_{t_0}^{v}({}_{gH}D_*^{v}\sum_{i=0}^{\infty} y_i(t)) &= J_{t_0}^{v}(\sum_{i=0}^{\infty}{}_{gH}D_*^{v}\, y_i(t))\\ &=\sum_{i,0}^{\infty} J_{t_0}^{v}({}_{gH}D_*^{v}y_i(t))=J_{t_0}^{v}\,F(t,y(t))\\ &=J_{t_0}^{v}({}_{gH}D_*^{v}y_0(t))\oplus\sum_{i=1}^{\infty} J_{t_0}^{v}({}_{gH}D_*^{v}y_i(t))\\ &=J_{t_0}^{v}\left[Ly(t)\oplus Ny(t)\right]\\ &=J_{t_0}^{v}\sum_{i=0}^{\infty} Ly_i(t)+J_{t_0}^{v}\sum_{i=0}^{\infty} A_i(t). \end{aligned}$$

Now by Lemma (2.1) we obtain $(y_0(t)\ominus y_0(t_0))\oplus\sum_{i=1}^{\infty}[y_i(t)\ominus y_i(t_0)]=J_{t_0}^{v}\sum_{i=0}^{\infty} Ly_i(t)\oplus J_{t_0}^{v}\sum_{i=0}^{\infty} A_i(t),$

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Thus by suppose that $y_i(t_0) = \tilde{0}$ we find

$y_0(t) = y_0(t_0)$

$y_i(t) = \tilde{0} \ominus (-1)J_{t_0}^{\nu} L_{y_{i-1}}(t) \ominus (-1)\, J_{t_0}^{\nu} A_{i-1}(t). \quad i = 1,2,\dots$

(II) Suppose that $y_i(t)$, $\forall i\ (i = 0,1,\dots)$ is differentiable as in Definition 2.3(ii).similar to proof of (i), we have

$$J_{t_0}^{\nu}({}_{gH}D_*^{\nu} y_0(t)) \oplus \sum_{i=1}^{\infty} J_{t_0}^{\nu}({}_{gH}D_*^{\nu} y_i(t)) = J_{t0}^{\nu}[Ly(t) \oplus Ny(t)]\ ,$$

Thus

$$((-1)y_0(t_0) \ominus (-1)y_0(t)) \oplus \sum_{i=1}^{\infty} [-y_i(t_0) \ominus (-1)y_i(t)] = J_{t_0}^{\nu} \sum_{i=0}^{\infty} Ly_i(t) \oplus J_{t_0}^{\nu} \sum_{i=0}^{\infty} A_i(t).$$

Therefor

$y_0(t) = y_0(t_0)$

$y_i(t) = \tilde{0} \ominus (-1)J_{t_0}^{\nu} L_{y_{i-1}}(t) \ominus (-1)\, J_{t_0}^{\nu} A_{i-1}(t). \quad i = 1,2,\dots$

(III) Suppose that i is even, $y_i(t)$ is differentiable according to Definition 2.3(i) and j is odd, $y_j(t)$ is differentiable as in Definition 2.3(ii). Thus

$$J_{t_0}^{\nu}({}_{gH}D_*^{\nu} y_0(t)) \oplus J_{t_0}^{\nu}({}_{gH}D_*^{\nu} y_1(t)) \oplus J_{t_0}^{\nu}({}_{gH}D_*^{\nu} y_2(t)) \oplus \dots = J_{t_0}^{\nu} \sum_{i=0}^{\infty} Ly_i(t) \oplus J_{t_0}^{\nu} \sum_{i=0}^{\infty} A_i(t)$$

$$(y_0(t) \ominus y_0(t_0)) \oplus ((-1)y_1(t_0) \ominus (-1)y_1(t)) \oplus (y_2(t) \ominus y_2(t_0)) \oplus \dots = J_{t_0}^{\nu} \sum_{j=0}^{\infty} Ly_j(t) \oplus J_{t_0}^{\nu} \sum_{j=0}^{\infty} A_j(t).$$

Now we give

$y_0(t) = y_0(t_0),$

$y_i(t) = \tilde{0} \oplus J_{t_0}^{\nu}\, Ly_{i-1}(t) \oplus J_{t_0}^{\nu}\ A_{i-1}(t)\ , \qquad i\ \text{is even}$

$y_j(t) = \tilde{0} \ominus (-1)J_{t_0}^{\nu} Ly_{j-1}(t) \ominus (-1)J_{t_0}^{\nu} A_{j-1}(t). \qquad j\ \text{is odd}$

Therefor the proof is completed for even and odd indexes.

4. ADOMIAN DECOMPOSITION METHOD FOR SOLVING FUZZY CAPUTO'S FRACTIONAL DIFFERENTIAL EQUATIONS

Convergence of the ADM to the exact solution is established for many kinds of problems. For instace, Fuzzy initial-value problems, Fuzzy linear systems, Boundary value problems, Partial differential equations and etc. (Abdelrazec & Pelinovsky, 2011; Adomian, 1988; Adomian, 1994; Abbaoui & Cherruault, 1994; Agarwal et al., 2010)

In this section we are going to show the uniformly convergence of the successive iterations in theorem 3.1. into $y(t)$ as an application of ADM. To this end, we need some Lemmas.

Lemma 4.1. Adomian's polynomials are bounded. It means $D(A_i(t),\tilde{0}) \leq M$, that D is Hausdorff metric.

Proof:

$$D(A_i(t),\tilde{0}) = D\left(\frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N(\sum_{j=0}^{\infty}\lambda^j y_j)\right]_{|\lambda=0},\tilde{0}\right)$$
$$= \frac{1}{i!}D\left(\frac{d^i}{d\lambda^i}\left[N(\sum_{j=0}^{\infty}\lambda^j y_j)\right]_{|\lambda=0},\tilde{0}\right)$$

$\frac{d^i}{d\lambda^i}\left[N(\sum_{j=0}^{\infty}\lambda^j y_j)\right]_{\lambda=0}$ is a polynomial that for $\lambda=0$ is bounded. Thus the proof is completed.

Remark 4.1. $(x-s)^{\nu-1}$ in fuzzy Riemann-Liouville integral is continuous and bounded. Thus we suppose that $\int_{t_0}^{t}|x-s|^{\nu-1}d_s \leq N$.

Theorem 4.1. Assume that Eq. (3.1) satisfies the following conditions

(i) In the linear terms for $i \geq 1$, $D(L_{y_i}(t), L_{y_{i-1}}(t)) \leq P$ and $1 < P < N^{-1}$ such that N is explained in Remark 4.1 .

(ii) In the nonlinear terms for $i \geq 1$, $D(A_i(t), A_{i-1}(t)) \leq M$, and $1 < M < P^{-1}$.

Then the successive iterations in theorem 3.1. are uniformly convergent to $y(t)$ on $[t_0, a]$.

Proof:

We prove the theorem for case (I). The proof of other cases is similar to case (I) that omitted here. By hypothesis, for case (I) we get

$$D(y_{n+1}(t),y_n(t)) = D\left[\frac{1}{\Gamma(\nu)}\int_{t_0}^{t}(x-s)^{\nu-1}(Ly_n(s)\oplus A_n(s))ds, \frac{1}{\Gamma(\nu)}\int_{t_0}^{t}(x-s)^{\nu-1}(Ly_{n-1}(s)\oplus A_{n-1}(s))ds)\right]$$
$$\leq \frac{1}{\Gamma(\nu)}\int_{t_0}^{t}D\left[(x-s)^{\nu-1}\{Ly_n(s)\oplus A_n(s)\},(x-s)^{\nu-1}\{Ly_{n-1}(s)\oplus A_{n-1}(s)\}\right]ds$$
$$\leq \frac{1}{\Gamma(\nu)}\int_{t_0}^{t}|x-s|^{\nu-1}D\{Ly_n(s)\oplus A_n(s),Ly_{n-1}(s)\oplus A_{n-1}(s)\}ds$$
$$\leq \frac{1}{\Gamma(\nu)}\sup_{t_0\leq t\leq a}D\{Ly_n(t)\oplus A_n(t),Ly_{n-1}(t)\oplus A_{n-1}(t)\}\int_{t_0}^{t}|x-s|^{\nu-1}ds$$
$$\leq \frac{N}{\Gamma(\nu)}\sup_{t_0\leq t\leq a}D(y_n(t),y_{n-1}(t)). \quad (7).$$

Thus we get

$$\sup_{t_0\leq t\leq a}D(y_{n+1}(t),y_n(t)) \leq \frac{N}{\Gamma(\nu)}\sup_{t_0\leq t\leq a}D(y_n(t),y_{n-1}(t)). \quad (8).$$

With this manner we have

$$\sup_{t_0\le t\le a} D(y_n(t),y_{n-1}(t)) \le \frac{N}{\Gamma(\nu)} \sup_{t_0\le t\le a} D(y_{n-1}(t),y_{n-2}(t))$$
$$\le \frac{N^2}{\Gamma(\nu)^2} \sup_{t_0\le t\le a} D(y_{n-2}(t),y_{n-3}(t)) \le \ldots \le \frac{N^{n-1}}{\Gamma(\nu)^{n-1}} \sup_{t_0\le t\le a} D(y_2(t),y_1(t)). \tag{9}$$

For $n=1$, we get

$$D(y_2(t),y_1(t)) = D\left[\frac{1}{\Gamma(\nu)}\int_{t_0}^{t} (x-s)^{\nu-1}(Ly_1(s)\oplus A_1(s))ds, \frac{1}{\Gamma(\nu)}\int_{t_0}^{t} (x-s)^{\nu-1}(Ly_0(s)\oplus A_0(s))ds\right]$$
$$\le \frac{1}{\Gamma(\nu)}\int_{t_0}^{t} D\left[(x-s)^{\nu-1}(Ly_1(s)\oplus A_1(s)),(x-s)^{\nu-1}(Ly_0(s)\oplus A_0(s))\right]ds$$
$$\le \frac{1}{\Gamma(\nu)}\int_{t_0}^{t} |x-s|^{\nu-1}\left(D[Ly_1(s),Ly_0(s)]\oplus D[A_1(s),A_0(s)]\right)ds, \tag{10}$$

From (4.4) and by hypothesis, we obtain

$$\sup_{t_0\le t\le a} D(y_2(t),y_1(t)) \le \frac{(P+M)N}{\Gamma(\nu)}. \tag{11}$$

If we substitute (4.5) to (4.3) we will get

$$\sup_{t_0\le t\le a} D(y_{n+1}(t),y_n(t)) \le \frac{N^{n-1}}{\Gamma^{n-1}(\nu)}\frac{(P+M)N}{\Gamma(\nu)}$$
$$\le \frac{N^n}{\Gamma^n(\nu)}(P+M).$$

By hypothesis and since $P \le P^n$ and $M \le M^n$, we can write

$$\lim_{n\to\infty} \sup_{t_0\le t\le a} D(y_{n+1}(t),y_0(t)) \le \lim_{n\to\infty} \frac{N^n}{\Gamma^n(\nu)}(P^n+M^n)$$
$$= \lim_{n\to\infty} \frac{(NP)^n+(MP)^n}{\Gamma^n(\nu)} = 0,$$

So this guarantee that the sequence $\{y_n(t)\}$ is convergence and if we denote $y(t) = \lim_{n\to\infty} y_n(t)$.

Then $y(t)$ satisfies Equation (3).

By reasoning similar to case (I) the uniformly convergence of sequence $\{y_i(t)\}$ to $y(t)$ is established.

5. EXAMPLE

We consider example has nonlinear part to illustrate the ADM.

Example: Let us consider the nonlinear FFDE

$$\begin{cases} \left({}_{gH}D_*^{0.5}\, t\right) = y^2(t) + (t-t^2), & t>0 \\ y(0) = [\alpha, 2-\alpha] \in \Re_F. \end{cases}$$

The exact ${}^{CF}[(i)-gH]$-differentiable solutions is $y(t)=t$.

From Formula (4), the approximate solutions expressed as

$$y_1(t)=\frac{1}{\Gamma(0.5)}\int_0^t (t-x)^{-0.5}dx = 1.12838t^{0.5},$$

$$y_2(t)=\frac{1}{\Gamma(0.5)}\int_0^t (t-x)^{-0.5}1.12838x^{0.5}dx = 1.27324t,$$

$$y_3(t)=\frac{1}{\Gamma(0.5)}\int_0^t (t-x)^{-0.5}1.27324\,xdx = 1.4367t^{1.5},$$

$$y_4(t)=\frac{1}{\Gamma(0.5)}\int_0^t (t-x)^{-0.5}1.4367x^{1.5}dx = 1.62114t^2,...$$

By ADM we have

$$y(t)=\sum_{i=0}^{\infty} y_i(t) = 1.12838t^{0.5} + 1.12838t + 1.4367t^{1.5} + 1.62114t^2 + ...$$

Figure 1 shows the exact and approximate solutions. With Comparison between them we can see a good and acceptable converging of ADM.

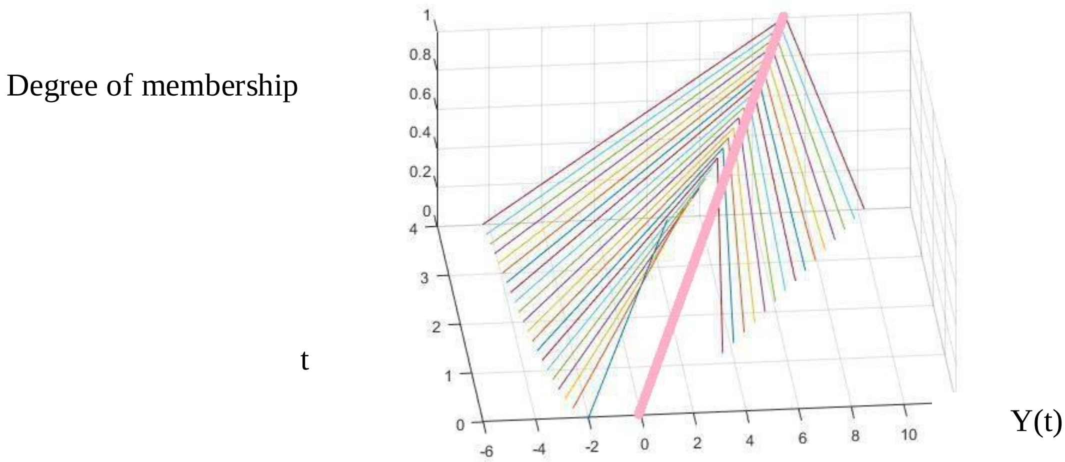

Figure 1: Comparison between exact and approximate solutions.

6. CONCLUSION

In this paper, we used the Adomian decomposition method (ADM) to obtain an approximation for solution of fuzzy Caputo fractional differential equations. This method is so powerful and efficient that it give approximations of higher accuracy. Also uniformly convergence of sequence $\{y_i(t)\}$ with various types of differentiability to the exact solution is proved.

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